

University Carlos III
Mathematics for Economics I.
Final Exam. January 17th 2025

Last Name:

Name:

Degree:

Group:

IMPORTANT

- **DURATION OF THE EXAM: 2h**
- Calculators are **NOT** allowed.
- **Scrap paper:** You may use the last two pages of this exam and the space behind this page.
- **Do NOT UNSTAPLE** the exam.
- You must show a valid ID to the professor.

Problem	Points
1	
2	
3	
4	
5	
6	
Total	

(1) Consider the set

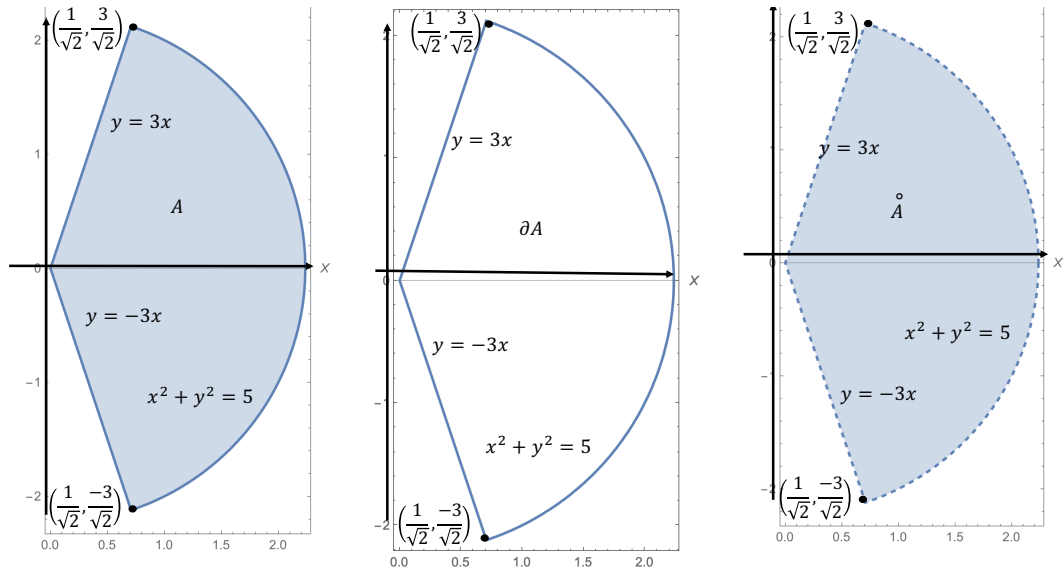
$$A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 5, -3x \leq y \leq 3x\}$$

and the function

$$f(x, y) = (2y - x)^3$$

- (a) **(20 points)** Sketch the graph of the set A , its boundary and its interior and justify if it is open, closed, bounded and/or compact.

Solution: The set A , its interior and its boundary are approximately as indicated in the picture.



And the closure coincides with A , $\bar{A} = A$. The functions $h_1(x, y) = y^2 + x^2 - 5$, $h_2(x, y) = y + 3x$ and $h_3(x, y) = y - 3x$ are continuous and $A = \{(x, y) \in \mathbb{R}^2 : h_1(x, y) \leq 0, h_2(x, y) \leq 0, h_3(x, y) \geq 0\}$. Hence, the set A is closed (Note also that $\partial A \subset A$). It is not open because $A \cap \partial A \neq \emptyset$. The set A is bounded. Therefore, the set A is compact.

- (b) **(10 points)** Prove that the set A is convex.

Solution: The set A is the intersection of 3 sets $A = A_1 \cap A_2 \cap A_3$ with

$$A_1 = \{(x, y) \in \mathbb{R}^2 : h_1(x, y) \leq 0\}$$

$$A_2 = \{(x, y) \in \mathbb{R}^2 : h_2(x, y) \geq 0\},$$

$$A_3 = \{(x, y) \in \mathbb{R}^2 : h_3(x, y) \leq 0\},$$

where the functions $h_1(x, y) = y^2 + x^2 - 5$, $h_2(x, y) = y + 3x$ and $h_3(x, y) = y - 3x$ have been defined in the previous part. The functions h_2 and h_3 are linear. Hence, the sets A_2 and A_3 are convex. We show next that the set A_1 is also convex. Consider the function $g(x, y) = y^2 + x^2 - 5$. Its Hessian matrix is

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

which is positive definite. Hence, the function g is convex and the set A_1 is convex. Since, A is the intersection of convex sets, it is also convex.

- (c) **(5 points)** State Weierstrass' Theorem. Determine if it is possible to apply Weierstrass' Theorem to the function f defined on A .

Solution: The set A is compact and the function $f(x, y) = (2y - x)^3$ is continuous. Hence, Weierstrass Theorem applies.

- (d) **(10 points)** Draw the level curves of f , indicating the direction of growth of the function.

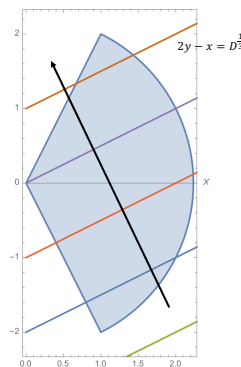
Solution: The level curves

$$f(x, y) = (2y - x)^3 = D$$

are straight lines of the form

$$y = \frac{x}{2} + \frac{D^{1/3}}{2}$$

Graphically,



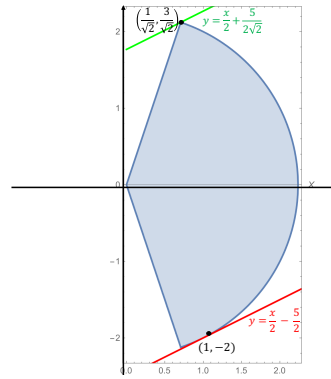
The black arrow represents the direction of growth of the function f .

- (e) **(20 points)** Using the level curves of f , determine (if they exist) the points of the set A where the function f attains its **global minimum** and/or a **global maximum** values on the set A .

Solution: Graphically, the minimum value is attained at the point (x_0, y_0) where the line $y = \frac{x}{2} + \frac{D^{1/3}}{2}$ is tangent to the graph of the curve $x^2 + y^2 = 5$. Let $y(x)$ be the function defined by the equation $x^2 + y^2 = 5$ near the point (x_0, y_0) . The slope of the line $y = \frac{x}{2} + \frac{D^{1/3}}{2}$ is $m = \frac{1}{2}$. Hence, $y'(x_0,) = \frac{1}{2}$. Taking derivatives, implicitly, we have

$$0 = 2x_0 + 2y_0y'(x_0) = 2x_0 + y_0$$

Thus, $y_0 = -2x_0$. Substituting this value in the equation $x_0^2 + y_0^2 = 5$, we obtain $5x_0^2 = 5$ or $x_0 = \pm 1$. Since $x_0 \geq 0$, we take $x_0 = 1$. The minimum value is attained at the point $(1, -2)$ and the minimum value of the function is $f(1, -2) = -125$.



Graphically, the maximum value is attained at the point where the curves $y = 3x$ and $y^2 + x^2 = 5$ intersect. That is, at the point where $10x^2 = 5$, $x > 0$. Hence, $x = \frac{1}{\sqrt{2}}$. That is, the maximum value is attained at the point $(\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}})$. The maximum value is $f(\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}}) = (\frac{5}{\sqrt{2}})^3$.

(2) Consider the function $f(x, y, z) = -x^3 - 2xy - y^3 + 2yz - z^2$ defined in $\mathbb{R}^3$.

- (a) **(15 points)** Determine D , the largest open set of $\mathbb{R}^3$ where the function f is strictly concave.

Solution: *We have*

$$\nabla f(x, y, z) = (-3x^2 - 2y, -2x - 3y^2 + 2z, 2y - 2z),$$

$$H(f)(x, y, z) = \begin{pmatrix} -6x & -2 & 0 \\ -2 & -6y & 2 \\ 0 & 2 & -2 \end{pmatrix}$$

We obtain $D_1 = -6x$, $D_2 = 36xy - 4$, $D_3 = -72xy + 24x + 8$. We require that

- $D_1 < 0$. Hence, $x > 0$.
- $D_2 > 0$. Hence, $y > \frac{1}{9x}$.
- $D_3 < 0$. Hence, $y > \frac{1}{9x} + \frac{1}{3}$.

Therefore,

$$D = \left\{ (x, y, z) \in \mathbb{R}^3 : x > 0, \quad y > \frac{1}{9x} + \frac{1}{3} \right\}$$

- (b) **(10 points)** Prove that the set D found in the previous part is convex.

Solution: *Let $D_1 = \{(x, y, z) \in \mathbb{R}^3 : x > 0, \}$, $D_2 = \{(x, y, z) \in \mathbb{R}^3 : y > \frac{1}{9x} + \frac{1}{3}\}$.*

We see that D_1 is convex. Consider the function $g(x) = \frac{1}{9x} + \frac{1}{3}$. We have $g''(x) = \frac{2}{9x^3} > 0$. Therefore, $g(x)$ is a convex function and D_2 is convex. We conclude that $D = D_1 \cap D_2$ is also convex.

(3) Consider the system of equations

$$\begin{aligned} ty - xy + xz &= 3 \\ t^2x + z &= 2 \end{aligned}$$

- (a) **(10 points)** Using the implicit function theorem, prove that the above system of equations determines implicitly two differentiable functions $y(x, t)$ and $z(x, t)$ in a neighborhood of the point $(x_0, y_0, z_0, t_0) = (0, 3, 2, 1)$.

Solution: We first remark that $(x_0, y_0, z_0, t_0) = (0, 3, 2, 1)$ is a solution of the system of equations. The functions $f_1(z, t) = ty - xy + xz - 3$ and $f_2(z, t) = t^2x + z - 2$ are of class C^∞ , because they are polynomials. We compute

$$\left| \frac{\partial(f_1, f_2)}{\partial(y, z)} \right|_{(z, t) = (0, 3, 2, 1)} = \begin{vmatrix} t - x & x \\ 0 & 1 \end{vmatrix}_{(z, t) = (0, 3, 2, 1)} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

By the implicit function theorem, the above system of equations determines implicitly two differentiable functions $y(x, t)$ and $z(x, t)$ in a neighborhood of the point $(x_0, y_0, z_0, t_0) = (0, 3, 2, 1)$.

- (b) **(15 points)** Compute

$$\frac{\partial y}{\partial x}(0, 1), \frac{\partial z}{\partial x}(0, 1), \frac{\partial y}{\partial t}(0, 1), \frac{\partial z}{\partial t}(0, 1)$$

Solution: Differentiating implicitly the system with respect to x ,

$$\begin{aligned} 0 &= t \frac{\partial y}{\partial x}(x, t) - x \frac{\partial y}{\partial x}(x, t) - y(x, t) + x \frac{\partial z}{\partial x}(x, t) + z(x, t) \\ 0 &= \frac{\partial z}{\partial x}(x, t) + t^2 \end{aligned}$$

We plug in the values $(x_0, y_0, z_0, t_0) = (0, 3, 2, 1)$ to obtain

$$\begin{aligned} 0 &= \frac{\partial y}{\partial x}(0, 1) - 1 \\ 0 &= \frac{\partial z}{\partial x}(0, 1) + 1 \end{aligned}$$

Therefore

$$\frac{\partial y}{\partial x}(0, 1) = 1, \quad \frac{\partial z}{\partial x}(0, 1) = -1$$

Differentiating now implicitly the system with respect to t ,

$$\begin{aligned} 0 &= t \frac{\partial y}{\partial t}(x, t) - x \frac{\partial y}{\partial t}(x, t) + y(x, t) + x \frac{\partial z}{\partial t}(x, t) \\ 0 &= \frac{\partial z}{\partial t}(x, t) + 2tx \end{aligned}$$

We plug in the values $(x_0, y_0, z_0, t_0) = (0, 3, 2, 1)$ to obtain

$$\begin{aligned} 0 &= \frac{\partial y}{\partial t}(0, 1) + 3 \\ 0 &= \frac{\partial z}{\partial t}(0, 1) \end{aligned}$$

Therefore

$$\frac{\partial y}{\partial t}(0, 1) = -3, \quad \frac{\partial z}{\partial t}(0, 1) = 0$$

- (c) **(10 points)** Compute Taylor's polynomial of order 1 of the functions $y(x, t)$ and $z(x, t)$ at the point $(x_0, t_0) = (0, 1)$.

Solution: *Taylor's polynomial of order 1 of the function $y(x, t)$ at the point $(x_0, t_0) = (0, 1)$ is*

$$P_1(x, t) = y(x_0, t_0) + \frac{\partial y}{\partial x}(x_0, t_0)(x - x_0) + \frac{\partial y}{\partial t}(x_0, t_0)(t - t_0) = 3 + x - 3(t - 1)$$

Taylor's polynomial of order 1 of the function $z(x, t)$ at the point $x_0 = 1$ is

$$P_2(x, t) = z(x_0, t_0) + \frac{\partial y}{\partial x}(x_0, t_0)(x - x_0) + \frac{\partial z}{\partial t}(x_0, t_0)(t - t_0) = 2 - x$$

- (d) **(5 points)** Use Taylor's polynomial of order 1 of the functions $y(x, t)$ and $z(x, t)$ at the point $(x_0, t_0) = (0, 1)$ to give an approximate value of $y(0.01, 0.99)$ and $z(0.01, 0.99)$.

Solution: *We have*

$$P_1(0.01, 0.99) = 3 + 0.01 + 3 \times 0.01 = 3.04$$

and

$$P_2(0.01, 0.99) = 2 - 0.01 = 1.99$$

(4) Consider the quadratic form

$$Q(x, y, z) = ax^2 + by^2 + 4xy + 2xz + 2yz + z^2$$

where $a, b \in \mathbb{R}$.

(a) **(5 points)** Compute the symmetric matrix A associated to Q .

Solution: *The associated matrix is*

$$A = \begin{pmatrix} a & 2 & 1 \\ 2 & b & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

(b) **(10 points)** Compute the leading principal minors D_1 , D_2 and D_3 of A .

Solution: *We have*

$$D_1 = a, \quad D_2 = \begin{vmatrix} a & 2 \\ 2 & b \end{vmatrix} = ab - 4, \quad D_3 = |A| = ab - a - b$$

(c) **(15 points)** For what values of $a \neq 0$ and $b \neq 0$ is the quadratic form Q positive definite?

Solution: *We need $D_1 > 0, D_2 > 0, D_3 > 0$. That is, $a > 0, ab > 4$ and $ab > a + b$. Hence, the quadratic form is positive definite for the values of (a, b) in region*

$$A_1 = \{(a, b) \in \mathbb{R}^2 : a > 0, b > 0, ab > 4, ab > a + b\}$$

For completeness, we argue that the condition $ab > 4$ is redundant. However, this is not necessary to get full credit in this part.

Suppose that $a > 0, b > 0$ and $ab > a + b$. Then, we have $a(b - 1) > b > 0$ and $b(a - 1) > a > 0$. So, we must have $a > 1, b > 1$. Given this, the condition $a(b - 1) > b$ may be rewritten as

$$a > \frac{b}{b-1}$$

For given $b > 1$, the minimum value of ab subject to the condition $a > \frac{b}{b-1}$ is attained for $a = \frac{b}{b-1}$. And for this value of a , we have that

$$ab = \frac{b^2}{b-1}$$

Now, the function $g(b) = \frac{b^2}{b-1}$ is convex for $b > 1$, the minimum value of g is attained at $b = 2$ and $g(2) = 4$. Hence,

$$ab > \frac{b^2}{b-1} \geq 4$$

We conclude that the quadratic form is positive definite for the values of (a, b) in region A_2 .

$$A_2 = \{(a, b) \in \mathbb{R}^2 : a > 1, b > 1, ab > a + b\}$$

(d) **(15 points)** For what values of $a \neq 0$ and $b \neq 0$ is the quadratic form Q negative definite?

Solution: *We need $D_1 < 0, D_2 > 0, D_3 < 0$. That is, $a < 0, ab > 4$ and $a + b > ab$. Note that $a < 0$ and $ab > 4$ imply that $b < 0$. But, then we have $a + b < 0$ which is not compatible with $a + b > ab > 4$.*

(e) **(15 points)** Classify the quadratic form Q for the values of $a = 0$, $b \in \mathbb{R}$.

Solution: For $a = 0$, we obtain that

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 2 & b & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

We consider the chain

$$D_1 = 0, \quad D_2 = -4, \quad D_3 = |A| = -b$$

We see that

- for $b \neq 0$, we have $D_2 < 0$, $D_3 \neq 0$ and the quadratic form is indefinite.
- For $a = b = 0$ the matrix is

$$\begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

We consider the chain

$$D_1 = a_{33} = 1 > 0, \quad D_2 = \begin{vmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1 < 0, \quad D_3 = |A| = 0$$

and the quadratic form is indefinite.

- (5) **(20 points)** Consider the function $f(x, y) : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and the functions $x(t, z), y(t, z) : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(x, y) = xy + y^2 \quad \text{and} \quad x(t, z) = t^2 + z, \quad y(t, z) = tz^2$$

And consider the composition $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $h(t, z) = f(x(t, z), y(t, z))$. Use the chain rule to compute

$$\frac{\partial h}{\partial t}(-1, 1), \quad \frac{\partial h}{\partial z}(-1, 1)$$

Solution:

$$x(-1, 1) = 2, \quad y(-1, 1) = -1$$

$$Df(x, y) = (y, x + 2y), \quad Df(2, -1) = (-1, 0)$$

Let $g(t, z) = (x(t, z), y(t, z)) = (t^2 + z, tz^2)$. We have

$$Dg(t, z) = \begin{pmatrix} 2t & 1 \\ z^2 & 2tz \end{pmatrix}, \quad Dg(-1, 1) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

By the chain rule,

$$\left(\frac{\partial h}{\partial t}(-1, 1) \quad \frac{\partial h}{\partial z}(-1, 1) \right) = Df(2, -1) Dg(-1, 1) = (-1, 0) \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} = (2, -1)$$

Therefore,

$$\frac{\partial h}{\partial t}(-1, 1) = 2, \quad \frac{\partial h}{\partial z}(-1, 1) = -1$$

(6) Consider the function

$$f(x, y) = 5x^4 + 3x^2 + 4xy + y^6 + 5y^2$$

- (a) **(5 points)** Write the gradient and the Hessian matrix of the function f at a point (x, y) .

Solution: *The gradient of the function f is*

$$\nabla f(x, y) = (20x^3 + 6x + 4y, 4x + 6y^5 + 10y)$$

The Hessian matrix is

$$Hf(x, y) = \begin{pmatrix} 60x^2 + 6 & 4 \\ 4 & 30y^4 + 10 \end{pmatrix}$$

- (b) **(10 points)** Write the polynomial of Taylor of order 1 of the function f at the point $p = (0, 1)$.

Solution: *At the point $p = (0, 1)$ we have $f(0, 1) = 6$ and*

$$\nabla f(0, 1) = (4, 16)$$

Thus,

$$P_1(x) = 6 + 4x + 16(y - 1) = 4x + 16y - 10$$

- (c) **(10 points)** Write the polynomial of Taylor of order 2 of the function f at the point $p = (0, 1)$.

Solution: *At the point $p = (0, 1)$ we get*

$$Hf(0, 1) = \begin{pmatrix} 6 & 4 \\ 4 & 40 \end{pmatrix}$$

Thus,

$$\begin{aligned} P_2(x) &= 6 + 4x + 16(y - 1) + \frac{1}{2} (6x^2 + 8x(y - 1) + 40(y - 1)^2) \\ &= 3x^2 + 4xy + 20y^2 - 24y + 10 \end{aligned}$$

- (d) **(10 points)** Prove that for any $(x, y) \in \mathbb{R}^2$ the following inequality holds

$$5x^4 + 3x^2 + 4xy + y^6 + 5y^2 \geq 4x + 16y - 10$$

and the inequality is strict except for $x = 0, y = 1$.

Solution: *The Hessian of f is*

$$Hf(x, y) = \begin{pmatrix} 60x^2 + 6 & 4 \\ 4 & 30y^4 + 10 \end{pmatrix}$$

We see that $D_1 = 60x^2 + 6 > 0$ and $D_2 = 1800x^2y^4 + 600x^2 + 180y^4 + 44 > 0$. Hence, the function f is strictly convex and we have that the tangent plane at any point is below the graph of the function. Therefore, for any $(x, y) \in \mathbb{R}^2$ we have

$$f(x, y) \geq P_1(x, y)$$

with equality only at the point $p = (0, 1)$.